EQUIVALENCES INDUCED BY ADJOINT FUNCTORS

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be two Grothendieck categories, $\mathbf{R} : \mathcal{A} \to \mathcal{B}$, $\mathbf{L} : \mathcal{B} \to \mathcal{A}$ a pair of adjoint functors, $S \in \mathcal{B}$ a generator, and $U = \mathbf{L}(S)$. U defines a hereditary torsion class in $\mathcal{A}$, which is carried by $\mathbf{L}$, under suitable hypotheses, into a hereditary torsion class in $\mathcal{B}$. We investigate necessary and sufficient conditions which assure that the functors $\mathbf{R}$ and $\mathbf{L}$ induce equivalences between the quotient categories of $\mathcal{A}$ and $\mathcal{B}$ modulo these torsion classes. Applications to generalized module categories, rings with local units and group graded rings are also given here.

INTRODUCTION

The motivation of the study of equivalences of categories comes from the fact that many "coincidences" arising in several areas of mathematics may be explained, at a general level, as consequences of a suitable equivalence. For the case of the Grothendieck categories, the original situation is the classical Morita theory, where the equivalences between two module categories over the rings R and S are described as the functors $\mathrm{Hom}(M, -)$ and $- \otimes_S M$, with M a progenerator of Mod-R and $S \cong \mathrm{End}_R(M)$. For this case, it may be seen that the study of the equivalences of categories is strongly linked to the one of the relationships between an R-module and its endomorphism ring.

Later on, various generalizations of the Morita theory were studied by many authors (see [5], [7], [8], [21]). If $\mathcal{A}$ is a locally finitely generated Grothendieck category, $M \in \mathcal{A}$ and $S = \mathrm{End}_{\mathcal{A}}(M)$, then, without any assumption on M, it was proved in [9] that $\mathrm{Hom}_{\mathcal{A}}(M, -)$ induces an equivalence between certain quotient categories of $\mathcal{A}$ and Mod-S respectively. More recently, in [4] it is proposed an approach of equivalences between two complete and cocomplete abelian categories $\mathcal{A}$ and $\mathcal{B}$, which is based on an arbitrary pair of adjoint functors $\mathbf{R} : \mathcal{A} \to \mathcal{B}$ and $\mathbf{L} : \mathcal{B} \to \mathcal{A}$. The results in [4] and [9] are among our main starting point here. The aim of the present paper is to investigate conditions for the application of the results given in [9]

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in the case of an arbitrary adjoint pair between Grothendieck categories, as in [4]. This aim was stimulated by the observation that, in several situations, it is preferable to replace the category Mod-S from [9], with one of modules over a ring without identity. Consequently, the usual Hom-functor may be replaced with one preserving some additional structures, for instance the grading as in [16], or local units as in [2]. In the same way, we also cover the case of the category of all additive contravariant functors $(\text{mod-}R)^{\text{op}} \to \mathcal{A}b$, R being a ring, which plays a central role in the representation theory of finite dimensional algebras (see [3] or [14]).

The paper is organized as follows. In Section 1 is presented the general theory, the main results being Proposition 1.13 and Theorem 1.18, which give necessary and sufficient conditions for certain subcategories of $\mathcal{A}$ and $\mathcal{B}$ to be equivalent, via some functors induced by $\mathbf{R}$ and $\mathbf{L}$. In Section 2 we are concerned with the category of modules over rings with several objects. We deduce equivalences for the case of the graded modules, the generalized R-modules, where R is an algebra over a field k, and the unital modules over a ring with local units.

Let us briefly present our general assumptions and notations. Throughout this paper rings are associative, and modules are right, unless otherwise stated. If $\mathcal{A}$ is a category, we shall write $A \in \mathcal{A}$ to indicate that A is an object of $\mathcal{A}$. If $A \in \mathcal{A}$, then $\mathcal{L}_{\mathcal{A}}(A)$ denotes the lattice of subobjects of A. We shall denote by $\operatorname{fg}\mathcal{A}$ and $\operatorname{fp}\mathcal{A}$ the full subcategories of $\mathcal{A}$ consisting of finitely generated, respectively finitely presented objects. The composition of two morphisms $f : A' \to A$ and $f' : A \to A''$ will be simply written as $f'f$. The same holds for functors. All functors between preadditive categories are additive. When we consider contravariant functors, this will be explicitly stated, otherwise the functors are covariant. If $\mathbf{f}$ is a functor, we shall denote by $\operatorname{Ker}\mathbf{f}$ the class of all objects which are carried into 0 by $\mathbf{f}$.

We refer to [17] or [19] for general facts about the theory of categories, and to [20] for the module theory and the torsion theory.

1. Localization and equivalences

1.1. General setting. Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories, and $\mathcal{A} \underset{\mathbf{L}}{\overset{\mathbf{R}}{\rightleftarrows}} \mathcal{B}$ a pair of adjoint functors. Let $S \in \mathcal{B}$ be a generator, and put $U = \mathbf{L}(S) \in \mathcal{A}$. Denote by

$$\sigma : \mathbf{1}_{\mathcal{B}} \to \mathbf{RL} \text{ and } \rho : \mathbf{LR} \to \mathbf{1}_{\mathcal{A}}$$

the unit, respectively the counit of the adjunction. Thus, for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$, we note the well-known relations

$$\mathbf{R}(\rho_A)\sigma_{\mathbf{R}(A)} = 1_{\mathbf{R}(A)} \text{ and } \rho_{\mathbf{L}(B)}\mathbf{L}(\sigma_B) = 1_{\mathbf{L}(B)}.$$

We know that $\mathbf{R}$ is limit preserving and $\mathbf{L}$ is colimit preserving, in particular, $\mathbf{R}$ is left exact and $\mathbf{L}$ is right exact. We define the full subcategories $\operatorname{Stat}(\mathbf{R})$, $\mathbf{L}(\mathcal{B})$, $\operatorname{Pres}[U]$, $\operatorname{Gen}[U]$ and $\boldsymbol{\sigma}[U]$ of $\mathcal{A}$, consisting of objects

$A \in \mathcal{A}$, for which ρ_A is an isomorphism, there is an object $B \in \mathcal{B}$ such that $A \cong \mathbf{L}(B)$, there exists an exact sequence $U^{(\Lambda')} \to U^{(\Lambda)} \to A \to 0$, there is an exact sequence $U^{(\Lambda)} \to A \to 0$, respectively there is an object $A' \in \mathrm{Gen}[U]$ such that A is a subobject of A'. We also consider the full subcategories $\mathrm{Adst}(\mathbf{R})$ and $\mathbf{R}(\mathcal{A})$ of $\mathcal{B}$, which contain those objects $B \in \mathcal{B}$, such that σ_B is an isomorphism, respectively there is an object $A \in \mathcal{A}$ with $B \cong \mathbf{R}(A)$.

Observe that we have the inclusions

$$\mathrm{Stat}(\mathbf{R}) \subseteq \mathbf{L}(\mathcal{B}) \subseteq \mathrm{Pres}[U] \subseteq \mathrm{Gen}[U] \subseteq \boldsymbol{\sigma}[U] \subseteq \mathcal{A},$$

$$\mathrm{Adst}(\mathbf{R}) \subseteq \mathbf{R}(\mathcal{A}) \subseteq \mathcal{B}.$$

In addition, note that an object A of $\mathcal{A}$ belongs to $\mathrm{Pres}[U]$ if and only if there is a short exact sequence $0 \to K \to U^{(\Lambda)} \to A \to 0$ with $K \in \mathrm{Gen}[U]$, and that the functors $\mathbf{R}$ and $\mathbf{L}$ induce an equivalence

$$\mathrm{Stat}(\mathbf{R}) \underset{\mathbf{L}}{\overset{\mathbf{R}}{\rightleftarrows}} \mathrm{Adst}(\mathbf{R}).$$

1.2. The torsion theory on $\mathcal{A}$ associated with U. It is easy to observe that $\mathrm{Gen}[U]$ is a *pretorsion class* on $\mathcal{A}$, that is, it is closed under quotients and direct sums. Its corresponding idempotent preradical (see [20, Chapter VI, Proposition 1.4]) is given by $\mathrm{Tr}_U : \mathcal{A} \to \mathcal{A}$, $\mathrm{Tr}_U(A) = \sum\{A' \in \mathcal{L}_{\mathcal{A}}(A) \mid A' \in \mathrm{Gen}[U]\}$. Clearly, $\mathrm{Tr}_U(A)$ is the greatest subobject of A belonging to $\mathrm{Gen}[U]$, and it is called the *trace of U in A*. By standard arguments, it may be verified that $\mathrm{Tr}_U(A) = \sum\{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_{\mathcal{A}}(U, A)\}$.

Define

$$\mathcal{F}_{\mathcal{A}} = \{F \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(A/\,\mathrm{Tr}_U(A), F) = 0 \text{ for all } A \in \mathcal{A}\},$$

$$\mathcal{T}_{\mathcal{A}} = \{T \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(T, F) = 0 \text{ for all } F \in \mathcal{F}_{\mathcal{A}}\}.$$

Therefore $(\mathcal{T}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is the torsion theory generated by the class $\{A/\,\mathrm{Tr}_U(A) \mid A \in \mathcal{A}\}$. As this class is closed under subobjects, it follows that the torsion theory is hereditary (as in [20, VI, Proposition 3.3]). To this torsion theory there corresponds a left exact (necessarily idempotent) radical [20, Chapter VI, Proposition 3.1], denoted here by $\mathbf{t}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$.

In addition, an object $A \in \mathcal{A}$ is called *U-distinguished*, if for any nonzero morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A', A)$, there is a morphism $h \in \mathrm{Hom}_{\mathcal{A}}(U, A')$ such that $fh \neq 0$. Then we have by [9, Proposition 1.2]

$$\mathcal{F}_{\mathcal{A}} = \{F \in \mathcal{A} \mid F \text{ is } U\text{-distinguished}\}.$$

1.3. The relationship between $\mathcal{T}_{\mathcal{A}}$, U and $\mathrm{Ker}\,\mathbf{R}$. In the sequel, an important role is played by the equality $\mathcal{T}_{\mathcal{A}} = \mathrm{Ker}\,\mathbf{R}$. Now, we proceed to find conditions for the validity of this equality. As a first step, observe that

$$\mathrm{Ker}\,\mathbf{R} = \mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U, -) \subseteq \mathcal{T}_{\mathcal{A}}.$$

Indeed, for all $A \in \mathcal{A}$ we have the natural isomorphism $\mathrm{Hom}_{\mathcal{B}}(S, \mathbf{R}(A)) \cong \mathrm{Hom}_{\mathcal{A}}(U, A)$, so $\mathrm{Hom}_{\mathcal{A}}(U, A) = 0$ if and only if $\mathbf{R}(A) = 0$, because S is a generator for $\mathcal{B}$. Moreover, let $T \in \mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U, -)$ and $F \in \mathcal{F}_{\mathcal{A}}$. Since F is

U-distinguished, we obtain $\mathrm{Hom}_{\mathcal{A}}(T,F)=0$, and this implies that $T\in\mathcal{T}_{\mathcal{A}}$. Therefore $\mathrm{Ker}\,\mathbf{R}=\mathcal{T}_{\mathcal{A}}$ if and only if $\mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U,-)=\mathcal{T}_{\mathcal{A}}$, or using the terminology of [9], U is a *CQF-3 object* of $\mathcal{A}$.

Further we have

$$\mathrm{Ker}\,\mathbf{R}=\{T\in\mathcal{A}\mid \mathrm{Hom}_{\mathcal{A}}(A,T)=0 \text{ for all } A\in\mathrm{Gen}[U]\}.$$

The inclusion of the class $\{T\in\mathcal{A}\mid \mathrm{Hom}_{\mathcal{A}}(A,T)=0, \text{ for all } A\in\mathrm{Gen}[U]\}$ in $\mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U,-)$ is obvious. Now, let $T\in\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(U,T)=0$, and let $A\in\mathrm{Gen}[U]$. Then there is an epimorphism $h:U^{(\Lambda)}\to A$, and consider the canonical injections $q_\lambda:U\to U^{(\Lambda)}, \lambda\in\Lambda$. If $f\in\mathrm{Hom}_{\mathcal{A}}(A,T)$, then $fhq_\lambda=0$ for all $\lambda\in\Lambda$, so $fh=0$, and $f=0$, h being an epimorphism. Thus $\mathrm{Hom}_{\mathcal{A}}(A,T)=0$, and the converse inclusion is proved. Note that the above equality shows that $\mathrm{Ker}\,\mathbf{R}$ is the pretorsion free class corresponding to $\mathrm{Gen}[U]$.

Recall that a class of objects of a Grothendieck category is called a *TTF-class* if it is both a torsion and a torsion free class. A hereditary torsion class is a TTF-class if and only if it is closed under products. We discus here, for later references, a particular case, namely when U is a projective object of $\mathcal{A}$. Then, obviously, $\mathrm{Ker}\,\mathbf{R}=\mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U,-)$ is a TTF-class. Moreover $\mathrm{Gen}[U]$ is closed under extensions, hence it is the corresponding torsion class, regarding $\mathrm{Ker}\,\mathbf{R}$ as torsion free class. Thus, by [20, Chapter VI, Proposition 2.3], Tr_U is a radical, so $\mathrm{Tr}_U(A/\mathrm{Tr}_U(A))=0$ for all $A\in\mathcal{A}$. Using this, we may prove the equality

$$\mathcal{F}_{\mathcal{A}}=\{F\in\mathcal{A}\mid \mathrm{Hom}_{\mathcal{A}}(T,F)=0 \text{ for all } T\in\mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U,-)\}.$$

Indeed, since $\mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U,-)\subseteq\mathcal{T}_{\mathcal{A}}$, the inclusion of $\mathcal{F}_{\mathcal{A}}$ in the class defined in the right hand side of the above equality always holds. Conversely, the generating class $\{A/\mathrm{Tr}_U(A)\mid A\in\mathcal{A}\}$ of the torsion theory $(\mathcal{T}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ is contained in $\mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(U,-)$. Therefore $\mathcal{T}_{\mathcal{A}}$ and $\mathrm{Ker}\,\mathbf{R}$ are equal, as torsion classes for two torsion theories with the same torsion free class, or equivalently, U is a CQF-3 object of $\mathcal{A}$.

Replacing the category $\mathcal{A}$ with the Grothendieck category $\boldsymbol{\sigma}[U]$, we can use weaker conditions, namely U to be Σ-quasiprojective instead of projective, or $\mathbf{R}$ to be exact only on short exact sequences with terms in $\boldsymbol{\sigma}[U]$. Of course, in this case $(\mathcal{T}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ is a torsion theory in $\boldsymbol{\sigma}[U]$. Throughout this paper we shall have in mind the possibility of replacing $\mathcal{A}$ with $\boldsymbol{\sigma}[U]$.

1.4. Lemma. *With the above notations, the following statements are true.*

a) $\mathbf{R}(\mathrm{Im}\,\rho_A)\cong\mathbf{R}(A)$ *for all* $A\in\mathcal{A}$, *and* $\mathrm{Im}\,\rho_A$ *is the smallest subobject of* A *which satisfies this property;*

b) $\mathrm{Im}\,\rho_A\in\mathrm{Gen}[U]$ *for all* $A\in\mathcal{A}$, *and*

$$\{A\in\mathcal{A}\mid \rho_A \text{ is an epimorphism}\}\subseteq\mathrm{Gen}[U].$$

Proof. a) Let A be an arbitrary object of $\mathcal{A}$. By the factorization of ρ_A trough its image we obtain the commutative diagram with exact row

$$\begin{tikzcd}
 & & \mathbf{LR}(A) \arrow[dl] \arrow[d, "\rho_A"] \\
0 \arrow[r] & \operatorname{Im}\rho_A \arrow[r] & A.
\end{tikzcd}$$

This induces a factorization

$$\mathbf{RLR}(A) \overset{\mathbf{R}(\rho_A)}{\longrightarrow} \mathbf{R}(A) = \mathbf{RLR}(A) \to \mathbf{R}(\operatorname{Im}\rho_A) \to \mathbf{R}(A),$$

where $\mathbf{R}(\operatorname{Im}\rho_A) \to \mathbf{R}(A)$ is a monomorphism, since $\mathbf{R}$ is left exact. But $\mathbf{R}(\rho_A)$ is an epimorphism, so $\mathbf{R}(\operatorname{Im}\rho_A) \to \mathbf{R}(A)$ is an epimorphism too, hence an isomorphism.

Let now A' be a subobject of A. If $\operatorname{Im}\rho_A \leq A'$, the left exactness of $\mathbf{R}$ implies $\mathbf{R}(\operatorname{Im}\rho_A) \leq \mathbf{R}(A') \leq \mathbf{R}(A)$, hence $\mathbf{R}(A') \cong \mathbf{R}(A)$. Conversely, if $\mathbf{R}(A') \cong \mathbf{R}(A)$, then the commutative diagram with exact bottom row

$$\begin{tikzcd}
 & \mathbf{LR}(A') \arrow[r, equal] \arrow[d, "\rho_{A'}"] & \mathbf{LR}(A) \arrow[d, "\rho_A"] \\
0 \arrow[r] & A' \arrow[r] & A
\end{tikzcd}$$

shows that ρ_A factors through A', or equivalently, $\operatorname{Im}\rho_A \leq A'$.

b) This is immediate, since $\mathbf{LR}(A) \in \operatorname{Gen}[U]$ for all $A \in \mathcal{A}$, and $\operatorname{Gen}[U]$ is closed under quotients. $\square$

1.5. Lemma. *The following statements are equivalent for the adjoint pair* $(\mathbf{R}, \mathbf{L})$.

(i) $\operatorname{Im}\rho_A = \operatorname{Tr}_U(A)$ *for all* $A \in \mathcal{A}$*;*

(ii) $\{A \in \mathcal{A} \mid \rho_A \text{ is an epimorphism}\} = \operatorname{Gen}[U]$

Moreover, if any of these conditions holds, then $\operatorname{Coker}\rho_A \in \mathcal{T}_{\mathcal{A}}$ *for all* $A \in \mathcal{A}$*.*

Proof. (i)⇒(ii). As we have already seen in Lemma 1.4, we must prove only the inclusion of $\operatorname{Gen}[U]$ in the class $\{A \in \mathcal{A} \mid \rho_A \text{ is an epimorphism}\}$. But assuming (i), this is immediate, since $A \in \operatorname{Gen}[U]$ if and only if $A = \operatorname{Tr}_U(A)$. Moreover, it is clear that $\operatorname{Coker}\rho_A = A/\operatorname{Tr}_U(A) \in \mathcal{T}_{\mathcal{A}}$.

(ii)⇒(i). Let $A' = \operatorname{Tr}_U(A)$. Then $\rho_{A'}$ is an epimorphism, and $\operatorname{Im}\rho_A \leq A'$. Thus the commutative diagram

$$\begin{tikzcd}
 & \mathbf{LR}(A') \arrow[r] \arrow[d, "\rho_{A'}"] & \mathbf{LR}(A) \arrow[dl] \arrow[d, "\rho_A"] \\
0 \arrow[r] & A' \arrow[r] & A
\end{tikzcd}$$

shows that $\mathbf{LR}(A) \to A'$ is an epimorphism, so $A' = \operatorname{Im}\rho_A$. $\square$

1.6. Proposition. *If* $\operatorname{Pres}[U] = \operatorname{Stat}(\mathbf{R})$*, then the following statements hold.*

a) $\sigma_{\mathbf{R}(A)}, \mathbf{R}(\rho_A), \mathbf{L}(\sigma_B), \rho_{\mathbf{L}(B)}$ *are isomorphisms for all* $A \in \mathcal{A}$ *and all* $B \in \mathcal{B}$*;*

b) $\operatorname{Pres}[U]$ *is a reflective subcategory of* $\mathcal{A}$ *with the reflector* $\mathbf{LR}$*;*

c) $\mathbf{R}(\mathcal{A})$ *is a coreflective subcategory of* $\mathcal{A}$ *with the coreflector* $\mathbf{RL}$*;*

d) $\operatorname{Ker}\rho_A \in \mathcal{T}_{\mathcal{A}}$ *for all* $A \in \mathcal{A}$*.*

Proof. According to [4, Theorem 1.6], the hypothesis $\operatorname{Pres}[U] = \operatorname{Stat}(\mathbf{R})$ is equivalent to $\mathbf{R}(\mathcal{A}) = \operatorname{Adst}(\mathbf{R})$.

a) Let $A \in \mathcal{A}$. Then $\mathbf{R}(A) \in \mathbf{R}(\mathcal{A})$, so $\sigma_{\mathbf{R}(A)}$ is an isomorphism, and $\mathbf{R}(\rho_A) = \sigma^{-1}_{\mathbf{R}(A)}$. Since $\mathbf{L}(\mathcal{B}) \in \operatorname{Pres}[U]$, what remains follows in a dual manner.

b) Recall that a full subcategory of a given category is called *(co)reflective* if the inclusion functor has a (left) right adjoint [17, Chapter V, Section 5]. (Note also that this terminology is not universally accepted. Some times a full subcategory is called reflective if the inclusion functor has a left adjoint, see for instance [20, p. 213]). Let now $A \in \operatorname{Pres}[U]$ and $A' \in \mathcal{A}$. Using the adjunction between $\mathbf{R}$ and $\mathbf{L}$ and a), we deduce the natural isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(A, \mathbf{LR}(A')) \cong \operatorname{Hom}_{\mathcal{A}}(\mathbf{LR}(A), \mathbf{LR}(A')) \cong$$
$$\operatorname{Hom}_{\mathcal{B}}(\mathbf{R}(A), \mathbf{RLR}(A')) \cong \operatorname{Hom}_{\mathcal{B}}(\mathbf{R}(A), \mathbf{R}(A')),$$

and

$$\operatorname{Hom}_{\mathcal{A}}(A, A') \cong \operatorname{Hom}_{\mathcal{A}}(\mathbf{LR}(A), A') \cong \operatorname{Hom}_{\mathcal{B}}(\mathbf{R}(A), \mathbf{R}(A')).$$

Consequently, $\mathbf{LR}$ is the right adjoint of the inclusion functor $\operatorname{Pres}[U] \to \mathcal{A}$.

c) This is the dual of b).

d) Applying the left exact functor $\mathbf{R}$ to the exact sequence

$$0 \to \operatorname{Ker}\rho_A \to \mathbf{LR}(A) \xrightarrow{\rho_A} A,$$

using a) and the inclusion $\operatorname{Ker}\mathbf{R} \subseteq \mathcal{T}_{\mathcal{A}}$, the assertion follows. $\square$

1.7. Limits and colimits in $\operatorname{Pres}[U]$**.** Assume the hypothesis of the previous proposition. Thus $\mathbf{LR} : \mathcal{A} \to \operatorname{Pres}[U]$ is limit preserving, and the inclusion functor $\operatorname{Pres}[U] \to \mathcal{A}$ is colimit preserving. In addition we know, by [17, Chapter V, duals of Proposition 5.1 and Proposition 5.2], that $\operatorname{Pres}[U]$ is a complete and cocomplete category. More precisely, let $\{A_\lambda \xrightarrow{\varphi_{\lambda,\lambda'}} A_{\lambda'} \mid \lambda, \lambda' \in \Lambda\}$ be a diagram in $\operatorname{Pres}[U]$. If $\{A \xrightarrow{\varphi_\lambda} A_\lambda \mid \lambda \in \Lambda\}$ is its limit in $\mathcal{A}$, then $\{\mathbf{LR}(A) \xrightarrow{\rho_A} A \xrightarrow{\varphi_\lambda} A_\lambda \mid \lambda \in \Lambda\}$ is its limit in $\operatorname{Pres}[U]$; if $\{A_\lambda \xrightarrow{\varphi_\lambda} A \mid \lambda \in \Lambda\}$ is its colimit in $\mathcal{A}$, then $A \cong \mathbf{LR}(A) \in \operatorname{Pres}[U]$, and it is its colimit in $\operatorname{Pres}[U]$ as well.

In particular, if $\operatorname{Pres}[U]$ is normal and conormal (which, by [17, Chapter I, Theorem 20.1] is equivalent to be abelian), then a morphism in $\operatorname{Pres}[U]$ is a monomorphism (epimorphism) in $\operatorname{Pres}[U]$ if and only if $\mathbf{LR}$ carries its kernel into the zero object of $\mathcal{A}$ (respectively, it is an epimorphism in $\mathcal{A}$). This situation happens, for instance, if $\mathbf{LR} : \mathcal{A} \to \operatorname{Pres}[U]$ is right exact

[17, V, dual of Proposition 5.3], this being actually equivalent to the right exactness of the functor $\mathbf{LR} : \mathcal{A} \to \mathcal{A}$, since the colimits in $\operatorname{Pres}[U]$ are computed exactly as in $\mathcal{A}$.

1.8. Filters corresponding to hereditary torsion classes. Let $\mathcal{B}$ be a Grothendieck category with a generator S, and let $\mathcal{T}$ be a hereditary torsion class on $\mathcal{B}$. Define $\mathcal{G} = \{I \in \mathcal{L}_{\mathcal{B}}(S) \mid S/I \in \mathcal{T}\}$. Using standard arguments, as in [20, Chapter VI, Proposition 4.2], we may see that $\mathcal{G}$ is a *filter* on the lattice $\mathcal{L}_{\mathcal{B}}(S)$ (that means that if $I_1 \in \mathcal{G}$, $I_2 \in \mathcal{L}_{\mathcal{B}}(S)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{G}$, and if $I_1, I_2 \in \mathcal{G}$, then $I_1 \cap I_2 \in \mathcal{G}$). Moreover, every torsion object $B \in \mathcal{T}$ may be regarded as a direct limit of a family of subobjects $\{B_\lambda \mid \lambda \in \Lambda\}$, such that $B_\lambda \cong S/I_\lambda$ for some $I_\lambda \in \mathcal{G}$, for all $\lambda \in \Lambda$.

Indeed, if B is of this form, then, clearly, $B \in \mathcal{T}$. Conversely, let $p : S^{(\Lambda)} \to B$ be an epimorphism, with $B \in \mathcal{T}$. Consider the canonical injections $q_\lambda : S \to S^{(\Lambda)}$, and let $B_\lambda = \operatorname{Im} pq_\lambda \leq B$. For all $\lambda \in \Lambda$ denote by p_λ the factorization of pq_λ through its image, and let $I_\lambda = \operatorname{Ker} p_\lambda$. We obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I_\lambda \arrow[r] & S \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"] & B_\lambda \arrow[r] \arrow[d] & 0 \\
 & & S^{(\Lambda)} \arrow[r, "p"] & B \arrow[r] & 0.
\end{tikzcd}$$

Obviously, $\varinjlim B_\lambda \leq B$. Since $S/I_\lambda \cong B_\lambda \in \mathcal{T}$, it follows $I_\lambda \in \mathcal{G}$. If $\pi : B \to B/\varinjlim B_\lambda$ denotes the canonical projection, then $\pi p q_\lambda = 0$ for all $\lambda \in \Lambda$, hence $\pi p = 0$, and $\pi = 0$, because p is an epimorphism. Therefore $B = \varinjlim B_\lambda$.

Note that, in the above situation, $\mathcal{T}$ is *generated* by $\{S/I \mid I \in \mathcal{G}\}$, that is, the corresponding torsion free class consists exactly of those objects $B \in \mathcal{B}$ for which $\operatorname{Hom}_{\mathcal{B}}(S/I, B) = 0$ for all $I \in \mathcal{G}$ (see also [20, Chapter VI, Section 2]).

1.9. The torsion theory in $\mathcal{B}$ associated with $\mathcal{T}_{\mathcal{A}}$. Assume that the object $\operatorname{Ker} \mathbf{L}(g)$ is $\mathcal{T}_{\mathcal{A}}$-torsion for every monomorphism g in $\mathcal{B}$. Then

$$\mathcal{T}_{\mathcal{B}} = \{B \in \mathcal{B} \mid \mathbf{L}(B) \in \mathcal{T}_{\mathcal{A}}\}$$

is a hereditary torsion class of objects of $\mathcal{B}$.

Without any additional assumptions, it is straightforward to check that $\mathcal{T}_{\mathcal{B}}$ is closed under quotients, extensions and direct sums. If $g : B' \to B$ is a monomorphism in $\mathcal{B}$ with $B \in \mathcal{T}_{\mathcal{B}}$, then $\operatorname{Ker} \mathbf{L}(g), \operatorname{Im} \mathbf{L}(g) \in \mathcal{T}_{\mathcal{A}}$. Thus the short exact sequence $0 \to \operatorname{Ker} \mathbf{L}(g) \to \mathbf{L}(B') \to \operatorname{Im} \mathbf{L}(g) \to 0$ shows that $\mathbf{L}(B') \in \mathcal{T}_{\mathcal{A}}$, hence $\mathcal{T}_{\mathcal{B}}$ is hereditary.

We denote by $\mathcal{F}_{\mathcal{B}}$ the corresponding torsion free class, and by $\mathbf{t}_{\mathcal{B}}$ the associated left exact (again necessarily idempotent) radical. In order to describe better this torsion theory, denote by IU the image of the induced morphism $\mathbf{L}(I) \to U$, for any subobject I of S. Clearly, $IU \leq U$, and put

$$\mathcal{G} = \{I \in \mathcal{L}_{\mathcal{B}}(S) \mid U/IU \in \mathcal{T}_{\mathcal{A}}\}.$$

Thus we have:

1.10. Lemma. *Suppose that* $\operatorname{Ker} \mathbf{L}(g) \in \mathcal{T}_{\mathcal{A}}$ *for any monomorphism* g *in* $\mathcal{B}$. *Then, the torsion theory* $(\mathcal{T}_{\mathcal{B}}, \mathcal{F}_{\mathcal{B}})$ *is generated by the set* $\{S/I \mid I \in \mathcal{G}\}$.

Proof. Using paragraph 1.8 it is enough to observe that

$$\mathcal{G} = \{I \in \mathcal{L}_{\mathcal{B}}(S) \mid S/I \in \mathcal{T}_{\mathcal{B}}\}.$$

Indeed, if $I \leq S$, then the short exact sequence $0 \to I \to S \to S/I \to 0$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \mathbf{L}(I) & \longrightarrow & \mathbf{L}(S) & \longrightarrow & \mathbf{L}(S/I) & \longrightarrow & 0 \\ & & \downarrow & & \| & & & & \\ 0 & \longrightarrow & UI & \longrightarrow & U & \longrightarrow & U/UI & \longrightarrow & 0, \end{array}$$

where $\mathbf{L}(I) \to IU$ is an epimorphism. Consequently, we obtain an epimorphism $\mathbf{L}(S/I) \to U/IU$, and the Ker-Coker lemma implies that it is actually an isomorphism. Hence $U/IU \in \mathcal{T}_{\mathcal{A}}$ is equivalent to $\mathbf{L}(S/I) \in \mathcal{T}_{\mathcal{A}}$, or to $S/I \in \mathcal{T}_{\mathcal{B}}$. □

1.11. Remark. The condition for $\operatorname{Ker} \mathbf{L}(g)$ to be torsion in $\mathcal{A}$ for every monomorphism g in $\mathcal{B}$ is the second important assumption of this section, the first being $\mathcal{T}_{\mathcal{A}} = \operatorname{Ker} \mathbf{R}$. While for this first condition, studied in paragraph 1.3, we have found some fairly general hypotheses such that it holds, now it seems more difficult to give such hypotheses, unless we consider particular situations.

1.12. Localization for the categories $\mathcal{A}$ and $\mathcal{B}$. By *localization* of an abelian category we understand an exact functor defined on it, which has a fully-faithful right adjoint. It is well known that, in the case of the Grothendieck categories, the kernel of this exact functor is a hereditary torsion class, and conversely, if a hereditary torsion class of a Grothendieck category is given, then it defines such a localization between the initial category and the so called *quotient category* modulo this torsion class. Moreover the quotient category is Grotendieck too. This is the reason why the more general concept of *localizing subcategory* coincides, in the case of the Grothendieck categories, with the one of hereditary torsion class.

Assume that $\operatorname{Ker} \mathbf{L}(g)$ is $\mathcal{T}_{\mathcal{A}}$-torsion for every monomorphism g in $\mathcal{B}$. Then $\mathcal{T}_{\mathcal{B}}$ is a hereditary torsion class, and we consider the quotient categories $\mathcal{C} = \mathcal{A}/\mathcal{T}_{\mathcal{A}}$, $\mathcal{D} = \mathcal{B}/\mathcal{T}_{\mathcal{B}}$ with the canonical functors

$$\mathcal{A} \underset{\mathbf{i}}{\overset{\mathbf{a}}{\rightleftarrows}} \mathcal{C} \text{ and } \mathcal{B} \underset{\mathbf{j}}{\overset{\mathbf{b}}{\rightleftarrows}} \mathcal{D},$$

where $\mathbf{a}, \mathbf{b}$ are exact, $\mathbf{i}, \mathbf{j}$ are fully-faithful right adjoints for $\mathbf{a}$, respectively $\mathbf{b}$, and $\mathcal{T}_{\mathcal{A}} = \operatorname{Ker} \mathbf{a}$, $\mathcal{T}_{\mathcal{B}} = \operatorname{Ker} \mathbf{b}$. We shall denote by

$$\nu : \mathbf{1}_{\mathcal{A}} \to \mathbf{ia} \text{ and } \mu : \mathbf{ai} \to \mathbf{1}_{\mathcal{C}},$$

$$\eta : \mathbf{1}_{\mathcal{B}} \to \mathbf{jb} \text{ and } \delta : \mathbf{bj} \to \mathbf{1}_{\mathcal{D}},$$

the units and the counits of these adjunctions. Then, μ and δ are natural isomorphisms, and ν_A, η_B have torsion kernel and cokernel relative to $\mathcal{T}_{\mathcal{A}}$, respectively $\mathcal{T}_{\mathcal{B}}$, for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$.

Obviously, the condition $\operatorname{Ker} \mathbf{L}(g) \in \mathcal{T}_{\mathcal{A}}$ for every monomorphism g in $\mathcal{B}$ is equivalent to the (left) exactness of the functor $\mathbf{aL} : \mathcal{B} \to \mathcal{C}$. Moreover, it implies $\mathcal{T}_{\mathcal{B}} = \operatorname{Ker} \mathbf{aL}$.

As usual, we identify the categories $\mathcal{C}$ and $\mathcal{D}$ with the full subcategories of $\mathcal{A}$ and $\mathcal{B}$ consisting of $\mathcal{T}_{\mathcal{A}}$, respectively $\mathcal{T}_{\mathcal{B}}$-closed objects. Recall that an object A of $\mathcal{A}$ is called *$\mathcal{T}_{\mathcal{A}}$-closed*, if it is torsion free, and every morphism $f \in \operatorname{Hom}_{\mathcal{A}}(A, A')$ with torsion kernel and cokernel is a section (that means, there is a morphism $f' \in \operatorname{Hom}_{\mathcal{A}}(A', A)$ such that $f'f = 1_A$) [19, Chapter 4, Lemma 7.9]. With this identification, the functors $\mathbf{i}$ and $\mathbf{j}$ become inclusions of $\mathcal{C}$ in $\mathcal{A}$ and $\mathcal{D}$ in $\mathcal{B}$ respectively. Clearly, $\mathcal{D} = \mathcal{B}$ if and only if $\mathcal{G} = \{S\}$.

1.13. Proposition. *If* $\mathbf{aL}$ *is exact, then the following statements are equivalent.*

(i) *The functor* $\mathbf{Ri} : \mathcal{C} \to \mathcal{B}$ *is fully-faithful;*

(ii) ρ_C *has torsion kernel and cokernel for all objects* C *of* $\mathcal{C}$*;*

(iii) $\mathcal{C} \underset{\mathbf{aL}}{\overset{\mathbf{Ri}}{\rightleftarrows}} \mathcal{D}$ *are equivalences inverse to each other.*

Proof. (i)$\Leftrightarrow$(ii). The functor $\mathbf{aL} : \mathcal{B} \to \mathcal{C}$ is the left adjoint of $\mathbf{Ri}$, and the counit of the adjunction is given by

$$\mathbf{aLRi}(C) \xrightarrow{\mathbf{a}(\rho_{\mathbf{i}(C)})} \mathbf{ai}(C) \xrightarrow{\mu_C} C,$$

for all $C \in \mathcal{C}$. Thus, according to [19, Chapter 1, Theorem 13.10], $\mathbf{Ri}$ is fully-faithful if and only if this counit is an isomorphism. Since μ_C is always an isomorphism, this is equivalent to $\mathbf{a}(\rho_C) = \mathbf{a}(\rho_{\mathbf{i}(C)})$ to be invertible, or to $\operatorname{Ker} \rho_C, \operatorname{Coker} \rho_C \in \mathcal{T}_{\mathcal{A}}$ for all $C \in \mathcal{C}$.

(i)$\Rightarrow$(iii). Since the functor $\mathbf{aL}$ is exact, and $\mathbf{Ri}$ is fully-faithful, [19, Chapter 4, Theorem 7.11] states that $\mathbf{Ri}$ induces an equivalence between $\mathcal{C}$ and $\mathcal{B}/\operatorname{Ker} \mathbf{aL}$. But it is clear that $\operatorname{Ker} \mathbf{aL} = \mathcal{T}_{\mathcal{B}}$, and (iii) follows.

(iii)$\Rightarrow$(i) is obvious. □

1.14. Corollary. *If* $\mathbf{aL}$ *is exact, and, in addition* $\operatorname{Pres}[U] = \operatorname{Stat}(\mathbf{R})$*, then the categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are equivalent.*

Proof. Let $C \in \mathcal{C}$. Using Proposition 1.6 d), we have $\operatorname{Ker} \rho_C \in \mathcal{T}_{\mathcal{A}}$. On the other hand, [4, Lemma 1.4] implies the equality $\operatorname{Gen}[U] = \{A \in \mathcal{A} \mid \rho_A \text{ is an epimorphism}\}$, so $\operatorname{Coker} \rho_C \in \mathcal{T}_{\mathcal{A}}$ by Lemma 1.5, and the condition (ii) of the previous proposition is satisfied. □

1.15. Remark. If $\mathbf{R} = \mathrm{Hom}_{\mathcal{A}}(U, -)$, $S = \mathrm{End}_R(U)$ and $\mathcal{B} = \text{Mod-}S$, then $\mathbf{a}(U)$ is a generator of $\mathcal{C}$ (see [9, Lemma 1.3]). Thus the functor

$$\mathrm{Hom}_{\mathcal{A}}(\mathbf{a}(U), -) : \mathcal{C} \to \text{Mod-}\,\mathrm{End}_{\mathcal{C}}(\mathbf{a}(U))$$

is fully-faithful by the Popescu-Gabriel theorem [20, X, Theorem 4.1]. Since $\mathrm{Hom}_{\mathcal{A}}(U, C) \cong \mathrm{Hom}_{\mathcal{A}}(\mathbf{a}(U), C)$ for all $C \in \mathcal{C}$, and $\mathrm{End}_{\mathcal{C}}(\mathbf{a}(U))$ is the ring of quotients of S with respect to a certain topology by [9, proof of Theorem 1.6], it follows that the functor $\mathrm{Hom}_{\mathcal{A}}(U, -) : \mathcal{C} \to \text{Mod-}S$ is fully-faithful too. Therefore Proposition 1.13 gives the equivalence stated in [9, Theorem 1.6], while Lemma 1.10 gives the Gabriel topology on S. Of course, the argument used to show that $\mathrm{End}_{\mathcal{C}}(\mathbf{a}(U))$ is the ring of quotients of S is an important part of the proof of [9, Theorem 1.6].

1.16. Equivalences when $\mathcal{T}_{\mathcal{A}}$ is a TTF-class. Assume that $\mathcal{T}_{\mathcal{A}}$ is a TTF-class and denote by $\mathcal{H}$ the corresponding torsion class and by $\mathbf{h} : \mathcal{A} \to \mathcal{A}$ the associated idempotent radical [20, Chapter VI, Proposition 2.3]. Consider the full subcategory of $\mathcal{A}$ consisting of the objects belonging simultaneously to $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{H}$, that is

$$\mathrm{GF}[U] = \{A \in \mathcal{A} \mid A \in \mathcal{F}_{\mathcal{A}} \text{ and } A \in \mathcal{H}\}.$$

Then, by [9, Proposition 2.1] there are equivalences of categories inverse to each other

$$\mathcal{C} \to \mathrm{GF}[U], C \mapsto \mathbf{h}(C) \text{ and } \mathrm{GF}[U] \to \mathcal{C}, A \mapsto \mathbf{a}(A).$$

If the equivalent conditions of Proposition 1.13 are satisfied, then composing these equivalence with those stated in (iii), it follows that $\mathrm{GF}[U]$ and $\mathcal{D}$ are equivalent as well.

As we have already mentioned in paragraph 1.3, it is clear that a particular case in which $\mathcal{T}_{\mathcal{A}}$ is a TTF-class occurs if U is a projective object of $\mathcal{A}$. In this case $\mathcal{H} = \mathrm{Gen}[U]$, so $\mathbf{h} = \mathrm{Tr}_U$, and the category $\mathrm{GF}[U]$ consists exactly of those objects of $\mathcal{A}$ which are both U-generated and U-distinguished.

Before proving the main result of this section, we need the following technical lemma.

1.17. Lemma. *If U is a CQF-3 object of $\mathcal{A}$, then*

a) $\mathcal{T}_{\mathcal{B}} = \mathrm{Ker}\,\mathbf{L}$*;*

b) *If, in addition, the functor* $\mathbf{aL} : \mathcal{B} \to \mathcal{C}$ *is exact, then* $\mathrm{Ker}\,\mathbf{LR}(f) \in \mathcal{T}_{\mathcal{A}}$ *for every morphism f in $\mathcal{A}$ with torsion kernel.*

Proof. a) The inclusion $\mathrm{Ker}\,\mathbf{L} \subseteq \mathcal{T}_{\mathcal{B}}$ is clear by the definition of $\mathcal{T}_{\mathcal{B}}$. Conversely, if $B \in \mathcal{T}_{\mathcal{B}}$, then $\mathbf{L}(B) \in \mathcal{T}_{\mathcal{A}}$, and $\mathbf{RL}(B) = 0$. Hence $\mathbf{LRL}(B) = 0$, and also $\mathbf{L}(B) = 0$, since $\mathbf{L}(\sigma_B) : \mathbf{L}(B) \to \mathbf{LRL}(B)$ is a monomorphism.

b) Let $f : A \to A''$ be a morphism in $\mathcal{A}$ with $A' = \mathrm{Ker}\, f$ belonging to $\mathcal{T}_{\mathcal{A}}$. Applying the left exact functor $\mathbf{R}$ to the exact sequence $0 \to A' \to A \xrightarrow{f} A''$, and having in mind that $\mathbf{R}(A') = 0$, we obtain a monomorphism $\mathbf{R}(f) : \mathbf{R}(A) \to \mathbf{R}(A'')$. Thus, our assumption about $\mathbf{L}$ assures that $\mathrm{Ker}\,\mathbf{LR}(f)$ is torsion in $\mathcal{A}$. $\square$

1.18. Theorem. *If U is a CQF-3 object of $\mathcal{A}$, and the functor $\mathbf{aL} : \mathcal{B} \to \mathcal{C}$ is exact then:*
a) *The following conditions are equivalent.*
 (i) $\mathrm{Pres}[U] = \mathrm{Stat}(\mathbf{R})$;
 (ii) $\mathbf{R}(\mathcal{A}) = \mathrm{Adst}(\mathbf{R})$;
 (iii) $\mathcal{C} \underset{\mathbf{aL}}{\overset{\mathbf{Ri}}{\rightleftarrows}} \mathcal{D}$ *are equivalences inverse to each other.*
b) *If the conditions in a) hold, then the following conditions are also equivalent.*
 (i) $\mathrm{Pres}[U]$ *is an abelian category;*
 (ii) $\mathbf{R}(\mathcal{A}) = \mathcal{D}$;
 (iii) $\mathcal{C} \underset{\mathbf{a}}{\overset{\mathbf{LR}}{\rightleftarrows}} \mathrm{Pres}[U]$ *are equivalences inverse to each other;*
 (iv) $\mathrm{Pres}[U]$ *is a Grothendieck category;*
 (v) $\mathbf{RL} \cong \mathbf{jb}$;
 (vi) *The functor* $\mathbf{RL} : \mathcal{B} \to \mathcal{B}$ *is left exact.*
 (vii) *The functor* $\mathbf{LR} : \mathcal{A} \to \mathcal{A}$ *is right exact.*

Proof. a) (i)⇔(ii) is given in [4, Theorem 1.6].
(i)⇒(iii) is just Corollary 1.14.
(iii)⇒(i). To prove this, it is enough to show that $\mathbf{R}(\mathrm{Ker}\,\rho_A) = 0$ for all $A \in \mathcal{A}$ (see [4, Theorem 1.6]). In our case, this is equivalent to $\mathrm{Ker}\,\rho_A \in \mathcal{T}_{\mathcal{A}}$.
Let then $A \in \mathcal{A}$. We construct the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathrm{Ker}\, f \arrow[d] & \mathrm{Ker}\,\rho_A \arrow[d] & \mathrm{Ker}\,\rho_{\mathbf{ia}(A)} \arrow[d] \\
0 \arrow[r] & \mathrm{Ker}\,\mathbf{LR}(\nu_A) \arrow[r] \arrow[d, "f"] & \mathbf{LR}(A) \arrow[r, "\mathbf{LR}(\nu_A)"] \arrow[d, "\rho_A"] & \mathbf{LRia}(A) \arrow[d, "\rho_{\mathbf{ia}(A)}"] \\
0 \arrow[r] & \mathrm{Ker}\,\nu_A \arrow[r] & A \arrow[r, "\nu_A"] & \mathbf{ia}(A),
\end{tikzcd}$$

where f is induced by the definition of the kernel. The Ker-Coker lemma gives an exact sequence $0 \to \mathrm{Ker}\, f \xrightarrow{f'} \mathrm{Ker}\,\rho_A \to \mathrm{Ker}\,\rho_{\mathbf{ia}(A)}$. Consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r, "f'"] \arrow[d, equal] & \mathrm{Ker}\,\rho_A \arrow[r] \arrow[d, equal] & \mathrm{Coker}\, f' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r, "f'"] & \mathrm{Ker}\,\rho_A \arrow[r] & \mathrm{Ker}\,\rho_{\mathbf{ia}(A)},
\end{tikzcd}$$

where the induced morphism $\operatorname{Coker} f' \to \operatorname{Ker}\rho_{\mathbf{ia}(A)}$ is a monomorphism. Now, $\operatorname{Ker}\nu_A \in \mathcal{T}_{\mathcal{A}}$, so $\operatorname{Ker}\mathbf{LR}(\nu_A) \in \mathcal{T}_{\mathcal{A}}$ by Lemma 1.17. Consequently, $\operatorname{Ker} f \in \mathcal{T}_{\mathcal{A}}$, as a subobject of $\operatorname{Ker}\mathbf{LR}(\nu_A)$. On the other hand, $\mathbf{ia}(A) \in \mathcal{C}$ and, using (iii) and Proposition 1.13, we deduce that $\operatorname{Ker}\rho_{\mathbf{ia}(A)} \in \mathcal{T}_{\mathcal{A}}$. Hence its subobject $\operatorname{Coker} f'$ belongs to $\mathcal{T}_{\mathcal{A}}$ too. Therefore $\operatorname{Ker}\rho_A \in \mathcal{T}_{\mathcal{A}}$, and (i) follows.

b) (i)⇒(ii) Let $A \in \mathcal{A}$. If $B \in \mathcal{T}_{\mathcal{B}}$ then, by Lemma 1.17, we have $\operatorname{Hom}_{\mathcal{B}}(B, \mathbf{R}(A)) \cong \operatorname{Hom}_{\mathcal{A}}(\mathbf{L}(B), A) = \operatorname{Hom}_{\mathcal{A}}(0, A) = 0$, so $\mathbf{R}(A) \in \mathcal{F}_{\mathcal{B}}$.

If $g : \mathbf{R}(A) \to B$ is a morphism in $\mathcal{B}$ with torsion kernel and cokernel, it is actually a monomorphism, since its kernel is 0, being both torsion and torsion free. Denote $B'' = \operatorname{Coker} g$. The short exact sequence

$$0 \to \mathbf{R}(A) \xrightarrow{g} B \to B'' \to 0$$

induces an exact sequence

$$0 \to \operatorname{Ker}\mathbf{L}(g) \to \mathbf{LR}(A) \xrightarrow{\mathbf{L}(g)} \mathbf{L}(B) \to 0,$$

where we have used that $\mathbf{L}(B'') = 0$. Since $\operatorname{Ker}\mathbf{L}(g) \in \mathcal{T}_{\mathcal{A}}$, we have $\mathbf{LR}(\operatorname{Ker}\mathbf{L}(g)) = 0$, hence $\mathbf{L}(g)$ is a monomorphism and an epimorphism in $\operatorname{Pres}[U]$. But $\operatorname{Pres}[U]$ is balanced, as it is abelian, so $\mathbf{L}(g)$ is an isomorphism in $\operatorname{Pres}[U]$. Since $\operatorname{Pres}[U]$ is a full subcategory of $\mathcal{A}$, it follows that $\mathbf{L}(g)$ is an isomorphism in $\mathcal{A}$ too. Hence

$$\begin{aligned}\mathbf{R}(\rho_A)\mathbf{R}(\mathbf{L}(g)^{-1})\sigma_B g &= \mathbf{R}(\rho_A)\mathbf{R}(\mathbf{L}(g)^{-1})\mathbf{R}(\mathbf{L}(g))\sigma_{\mathbf{R}(A)} \\ &= \mathbf{R}(\rho_A)\sigma_{\mathbf{R}(A)} = 1_{\mathbf{R}(A)},\end{aligned}$$

and g is a section, which means $\mathbf{R}(A)$ is $\mathcal{T}_{\mathcal{B}}$-closed. Therefore $\mathbf{R}(\mathcal{A}) \subseteq \mathcal{D}$.

Now, let $B \in \mathcal{D}$. Consider the exact sequence

$$0 \to \operatorname{Ker}\sigma_B \to B \xrightarrow{\sigma_B} \mathbf{RL}(B) \to \operatorname{Coker}\sigma_B \to 0.$$

By Proposition 1.6, $\mathbf{L}(\sigma_B)$ is an isomorphism, hence $\mathbf{L}(\operatorname{Coker}\sigma_B) = 0$, and $\operatorname{Coker}\sigma_B \in \mathcal{T}_{\mathcal{B}}$. On the other hand, the composite morphism

$$\mathbf{L}(\operatorname{Ker}\sigma_B) \to \mathbf{L}(B) \xrightarrow{\mathbf{L}(\sigma_B)} \mathbf{LRL}(B)$$

is zero, implying that $\mathbf{L}(\operatorname{Ker}\sigma_B) \to \mathbf{L}(B)$ is the zero morphism. Thus $\mathbf{L}(\operatorname{Ker}\sigma_B) = \operatorname{Ker}(\mathbf{L}(\operatorname{Ker}\sigma_B) \to \mathbf{L}(B))$ belongs to $\mathcal{T}_{\mathcal{A}}$, and by the definition of $\mathcal{T}_{\mathcal{B}}$, $\operatorname{Ker}\sigma_B \in \mathcal{T}_{\mathcal{B}}$. Since B is $\mathcal{T}_{\mathcal{B}}$-closed, it follows that σ_B is a section, in particular a monomorphism, so B is a direct summand of $\mathbf{RL}(B)$. Let B' be the complement of B in $\mathbf{RL}(B)$. Since the additive functors preserve the exactness of the split short exact sequences, we obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & B' \arrow[r] \arrow[d, "\sigma_{B'}"] & \mathbf{RL}(B) \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "\sigma_B"] & 0 \\ 0 \arrow[r] & \mathbf{RL}(B') \arrow[r] & \mathbf{RLRL}(B) \arrow[r] & \mathbf{RL}(B) \arrow[r] & 0. \end{tikzcd}$$

Consequently, σ_B is an epimorphism. But we have seen that it is also a monomorphism, hence $B \cong \mathbf{RL}(B) \in \mathbf{R}(\mathcal{A})$.

(ii)⇒(iii). We have the equivalences

$$\mathrm{Pres}[U] \underset{\mathbf{L}}{\overset{\mathbf{R}}{\rightleftarrows}} \mathbf{R}(\mathcal{A}) \quad \text{and} \quad \mathcal{D} \underset{\mathbf{Ri}}{\overset{\mathbf{aL}}{\rightleftarrows}} \mathcal{C}.$$

Composing them, and having in mind that $\mathbf{R}(\mathcal{A}) = \mathcal{D}$ and $\mathbf{LR}(A) \cong A$ for all $A \in \mathrm{Pres}[U]$, we obtain (iii).

(iii)⇒(iv) and (iv)⇒(i) are obvious.

(ii)⇒(v). If (ii) holds, then $\mathbf{RL} : \mathcal{B} \to \mathcal{B}$ factors through the inclusion $\mathbf{j} : \mathcal{D} \to \mathcal{B}$, and this factorization is isomorphic to $\mathbf{b}$, as left adjoints of $\mathbf{j}$.

(v)⇒(vi) is immediate, since $\mathbf{b}$ is exact and $\mathbf{j}$ is left exact.

(vi)⇒(ii). The argument is similar to (i)⇒(ii), with the following modification. If $0 \to \mathbf{R}(A) \xrightarrow{g} B \to B'' \to 0$ is the above considered short exact sequence with $B'' \in \mathcal{T}_{\mathcal{B}}$, then, applying the left exact functor $\mathbf{RL}$, and keeping in mind that Lemma 1.17 implies $\mathbf{RL}(B'') = 0$, we deduce that $\mathbf{RL}(g)$ is an isomorphism.

(iii)⇒(vii). We have the (not necessarily commutative) diagram of categories and functors

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, shift left, "\mathbf{LR}"] \arrow[dr, shift left, "\mathbf{a}"] & \\
\mathrm{Pres}[U] \arrow[ur, shift left] \arrow[rr, shift left, "\mathbf{a}"] & & \mathcal{C}, \arrow[ul, shift left, "\mathbf{i}"] \arrow[ll, shift left, "\mathbf{LR}"]
\end{tikzcd}$$

where the functors on the bottom row are equivalences, and $\mathrm{Pres}[U] \to \mathcal{A}$ being the inclusion functor. This diagram shows that the functor $\mathbf{LR} : \mathcal{A} \to \mathrm{Pres}[U]$ has a right adjoint, namely $\mathbf{ia}$. Therefore it is right exact, and by the final observation of paragraph 1.7, $\mathbf{LR} : \mathcal{A} \to \mathcal{A}$ is right exact too.

(vii)⇒(i) is the dual of [17, Chapter V, Proposition 5.3]. □

1.19. Corollary. *If the functors $\mathbf{R}$ and $\mathbf{aL}$ are exact, and $U^{(\Lambda)} \in \mathrm{Stat}(\mathbf{R})$ for every set Λ, then $\mathbf{R}$ restricts to the following equivalences of categories*

a) $\mathcal{C} \longrightarrow \mathcal{D}$ *with inverse* $\mathcal{D} \xrightarrow{\mathbf{aL}} \mathcal{C}$*;*

b) $\mathrm{Pres}[U] \longrightarrow \mathcal{D}$ *with inverse* $\mathcal{D} \xrightarrow{\mathbf{L}} \mathrm{Pres}[U]$*;*

c) $\mathrm{GF}[U] \longrightarrow \mathcal{D}$ *with inverse* $\mathcal{D} \longrightarrow \mathrm{GF}[U]$*,* $B \mapsto \mathbf{L}(B)/\mathbf{t}_{\mathcal{A}}(\mathbf{L}(B))$*.*

Proof. It is easy to see that, with the hypoteses of this corollary, $\mathrm{Pres}[U] = \mathrm{Stat}(\mathbf{R})$ and $\mathbf{LR} : \mathcal{A} \to \mathcal{A}$ is right exact. Moreover, [4, Lemma 1.4] implies $\mathrm{Gen}[U] = \{A \in \mathcal{A} \mid \rho_A \text{ is an epimorphism}\}$, so $\mathrm{Tr}_U(A) = \mathrm{Im}\,\rho_A$ for all $A \in \mathcal{A}$, according to Lemma 1.5. Applying $\mathbf{R}$ to the exact sequence $0 \to \mathrm{Im}\,\rho_A \to A \to A/\mathrm{Im}\,\rho_A \to 0$, and using the isomorphism $\mathbf{R}(\mathrm{Im}\,\rho_A) \cong \mathbf{R}(A)$ stated in Lemma 1.4, we infer $A/\mathrm{Tr}_U(A) \in \mathrm{Ker}\,\mathbf{R}$, for all $A \in \mathcal{A}$. Since $\mathrm{Ker}\,\mathbf{R}$ is a TTF-class, the same argument as in paragraph 1.3 shows that U is a CQF-3 object of $\mathcal{A}$. Therefore a) and b) follow by Theorem 1.18. On the other hand, by paragraph 1.16, $\mathrm{GF}[U] \underset{\mathbf{h}}{\overset{\mathbf{a}}{\rightleftarrows}} \mathcal{C}$, are equivalences inverse to each other, where the meaning of the symbol $\mathbf{h}$ is the same as there. Moreover, if $A \in \mathrm{GF}[U]$ then it is torsion free, so ν_A is a monomorphism. Applying $\mathbf{R}$ to the short exact sequence

$$0 \to A \xrightarrow{\nu_A} \mathbf{ia}(A) \to \operatorname{Coker} \nu_A \to 0,$$

and having in mind that $\mathbf{R}(\operatorname{Coker} \nu_A) = 0$, we deduce $\mathbf{R}(A) \cong \mathbf{Ria}(A)$, so the functor $\mathrm{GF}[U] \to \mathcal{D}$ is the restriction of $\mathbf{R}$. Since $\mathbf{LR}$ is right exact and $\mathbf{LR}(\mathbf{t}_{\mathcal{A}}(A)) = 0$ for all $A \in \mathcal{A}$, it follows that $A \cong \mathbf{LR}(A) \cong \mathbf{LR}(A/\mathbf{t}_{\mathcal{A}}(A))$ for all $A \in \mathrm{Pres}[U]$. Moreover, composing the equivalences from a) and b), it follows that $\mathrm{Pres}[U]$ and $\mathrm{GF}[U]$ are also equivalent via the functors which map $A \mapsto \mathbf{ha}(A), A \in \mathrm{Pres}[U]$ and $F \mapsto \mathbf{LR}(F), F \in \mathrm{GF}[U]$. But, for all $A \in \mathrm{Pres}[U]$, $A/\mathbf{t}_{\mathcal{A}}(A) \in \mathrm{GF}[U]$, since $A/\mathbf{t}_{\mathcal{A}}(A) \in \mathrm{Gen}[U] \cap \mathcal{F}_{\mathcal{A}}$ and $\mathrm{Gen}[U] \subseteq \mathcal{H}$. Hence $\mathbf{ha} : \mathrm{Pres}[U] \to \mathrm{GF}[U]$ is isomorphic to the functor which carries A to $A/\mathbf{t}_{\mathcal{A}}(A)$. Consequently, the inverse of the restricton $\mathbf{R} : \mathrm{GF}[U] \to \mathcal{D}$ has the form indicated in c). □

1.20. Proposition. *If the functors* $\mathbf{R}$ *and* $\mathbf{aL}$ *are exact,* $U^{(\Lambda)} \in \mathrm{Stat}(\mathbf{R})$ *for every set* Λ*, and* U *is a subgenerator of* $\mathcal{A}$*, then the following statements are equivalent.*

(i) $\mathrm{GF}[U] \subseteq \mathrm{Pres}[U]$*;*

(ii) U *is a generator of* $\mathcal{A}$*;*

(iii) $\mathrm{Pres}[U] \subseteq \mathrm{GF}[U]$.

Moreover, if these conditions are satisfied, then the categories $\mathrm{Pres}[U]$*,* $\mathrm{GF}[U]$*,* $\mathcal{C}$*,* $\mathcal{A}$ *are equal, and* $\mathcal{T}_{\mathcal{A}} = \{0\}$.

Proof. A slight modification of [8, Proposition 1.5] shows the equivalence of the conditions (i), (ii) and (iii). Now, the last assertion is obvious.

□

1.21. Remark. a) Proposition 1.20 may be applied for an arbitrary U, replacing $\mathcal{A}$ with $\boldsymbol{\sigma}[U]$, where U is a subgenerator.

b) If R is an arbitrary ring with identity, $\mathcal{A} = \text{Mod-}R$, $\mathbf{R} = \mathrm{Hom}_R(U, -)$, $S = \mathrm{End}_{\mathcal{A}}(U)$, $\mathcal{B} = \text{Mod-}S$ and U is Σ-quasiprojective, then $U^{(\Lambda)} \in \mathrm{Stat}(\mathbf{R})$ for every set Λ [12, Theorem 2.1], so Corollary 1.19 gives the equivalences stated in [7, Theorem 1.3] and [8, Theorem 1.3], while Lemma 1.10 gives again the Gabriel topology on S. Clearly, by Proposition 1.20, $\mathcal{A} = \mathcal{C}$ if and only if U is a generator of $\mathcal{A}$.

2. Modules over small preadditive categories

2.1. Right pointed adjoint pair. Let $\mathcal{Y}$ be a (skeletally) small preadditive category. By a *(right) module over* $\mathcal{Y}$ (or simply $\mathcal{Y}$*-module*) we understand an additive contravariant functor $\mathcal{Y}^{\mathrm{op}} \to \mathcal{A}b$. The $\mathcal{Y}$-modules together with the natural transformations between them form a Grothendieck category, denoted here by $\text{Mod-}\mathcal{Y}$, the limits and the colimits being computed pointwise. Recall that an object A in a Grothendieck category $\mathcal{A}$ is called *finitely generated (presented)* if the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ commutes with direct unions (limits). It is well-known that the Yoneda functor $\mathcal{Y} \to \text{Mod-}\mathcal{Y}$, $Y \mapsto \text{Hom}_{\mathcal{Y}}(-, Y)$ is an embedding, hence the category $\mathcal{Y}$ may be identified with a full subcategory of Mod-$\mathcal{Y}$, consisting of finitely generated projective objects. Moreover these objects form a set of generators for Mod-$\mathcal{Y}$, what means that the category Mod-$\mathcal{Y}$ is *locally finitely generated*.

A functor $\mathbf{f} : \mathcal{Y} \to \mathcal{A}$, where $\mathcal{A}$ is an arbitrary Grothendieck category, induces an unique (up a natural isomorphism) colimits preserving functor $\mathbf{L} : \text{Mod-}\mathcal{Y} \to \mathcal{A}$, such that, with the above identification, $\mathbf{L}(Y) = \mathbf{f}(Y)$. Let $S = \bigoplus_{Y \in \mathcal{Y}} Y$ and $U = \mathbf{L}(S)$. The functor $\mathbf{L}$ has a right adjoint, namely $\mathbf{R} : \mathcal{A} \to \text{Mod-}\mathcal{Y}$, given by

$$\mathbf{R}(A) = \text{Hom}_{\mathcal{A}}(\mathbf{f}(-), A).$$

We shall say that the adjoint pair $(\mathbf{R}, \mathbf{L})$ is *induced by* $\mathbf{f}$. If we suppose in addition that $\mathbf{f}(Y)$ is a projective object of $\mathcal{A}$ for all $Y \in \mathcal{Y}$, then $\mathbf{R}$ is an exact functor, as it may be seen from its definition. Then U is projective, hence $\mathcal{T}_{\mathcal{A}} = \text{Ker}\,\mathbf{R} = \text{Ker}\,\text{Hom}_{\mathcal{A}}(U, -)$ by paragraph 1.3, where $\mathcal{T}_{\mathcal{A}}$ denotes the (hereditary) torsion class associated with U, which assures that U is a CQF-3 object.

We say that the adjoint pair $(\mathbf{R}, \mathbf{L})$ is *right pointed* if it is induced by a fully-faithful functor $\mathbf{f}$. If this is the case, then it may be easily seen that $\mathbf{RL}(Y) \cong Y$, for all $Y \in \mathcal{Y}$. Conversely, if $\sigma_Y : Y \to \mathbf{RL}(Y)$ is an isomorphism for all $Y \in \mathcal{Y}$, where σ denotes, as in paragraph 1.1, the unit of the adjunction between $\mathbf{R}$ and $\mathbf{L}$, then $\rho_{\mathbf{f}(Y)} = \rho_{\mathbf{L}(Y)}$ is an isomorphism too. Therefore the preadditive categories $\mathcal{Y}$ and $\mathbf{f}(\mathcal{Y})$ are equivalent, hence $\mathbf{f}$ is fully-faithful. Note also that the definition of right pointed adjoint pair given here agrees with the one imposed in [4, Section 2].

Assume now $(\mathbf{R}, \mathbf{L})$ is right pointed, and $\mathbf{f}(Y)$ is projective in $\mathcal{A}$ for all $Y \in \mathcal{Y}$, and let Q be a injective cogenerator of $(\mathcal{T}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$, thus $T \in \mathcal{T}_{\mathcal{A}}$ if and only if $\text{Hom}_{\mathcal{A}}(T, Q) = 0$. By standard arguments, like in [20, Chapter I, proof of Propositions 10.3 and 10.6], it may be verified that $\text{Ker}(\mathbf{L}(B') \to \mathbf{L}(B)) \in \mathcal{T}_{\mathcal{A}}$ for every short exact sequence $0 \to B' \to B \to B'' \to 0$ in $\mathcal{B}$, if and only if $\mathbf{R}(Q)$ is injective relative to this exact sequence. Fix now an object Y of $\mathcal{Y}$. We may show first, as in [12, Lemma 1.1], that $\sigma_I : I \to \mathbf{RL}(I)$ is an isomorphism for all finitely generated subobjects I of Y, further as in [8, proof of Theorem 1.3] that $\text{Ker}(\mathbf{L}(I) \to \mathbf{L}(Y)) \in \mathcal{T}_{\mathcal{A}}$ for all finitely generated subobjects I of Y. As Mod-$\mathcal{Y}$ is locally finitely generated, it follows by [10, paragraph 1.4], that every subobject I of Y is the direct union of its finitely generated subobjects. This implies (see again [8, proof of Theorem 1.3]) that $\text{Ker}(\mathbf{L}(I) \to \mathbf{L}(Y)) \in \mathcal{T}_{\mathcal{A}}$, for all $I \in \mathcal{L}_{\mathcal{B}}(Y)$. Consequently, $\mathbf{R}(Q)$ is injective relative to any exact sequence of the form $0 \to I \to Y \to Y/I \to 0$, with $Y \in \mathcal{Y}$. Furthermore this is equivalent, according to [13, Lemma 1], to the fact that $\mathbf{R}(Q)$ is an injective object of Mod-$\mathcal{Y}$. Finally, the interested fact for us is that $\text{Ker}(\mathbf{L}(B') \to \mathbf{L}(B)) \in \mathcal{T}_{\mathcal{A}}$, for all monomorphisms $B' \to B$ in $\mathcal{B}$. By paragraph 1.9 the class $\mathcal{T}_{\mathcal{B}}$ is a hereditary torsion class of objects of $\mathcal{B}$, and $\mathcal{T}_{\mathcal{B}} = \text{Ker}\,\mathbf{L}$, as in Lemma 1.17.

2.2. Gabriel topologies on $\mathcal{Y}$. A *Gabriel topology* of $\mathcal{B} = \text{Mod-}\mathcal{Y}$, following [10, paragraph 2.1], is a family $\mathcal{G} = \bigcup_{Y\in\mathcal{Y}} \mathcal{G}_Y$, where $\mathcal{G}_Y \subseteq \mathcal{L}_{\mathcal{B}}(H_Y)$, $Y \in \mathcal{Y}$ satisfying the following axioms
T1) $Y \in \mathcal{G}_Y$;
T2) if $I \in \mathcal{G}_Y$ and $g \in \text{Hom}_{\mathcal{B}}(Y', Y)$, then $g^{-1}(I) \in \mathcal{G}_{Y'}$;
T3) if $I_1, I_2 \in \mathcal{L}(Y)$ for $Y \in \mathcal{Y}$, and $I_1 \in \mathcal{G}_Y$, such that $g^{-1}(I_2) \in \mathcal{G}_{Y'}$ for any $g \in \text{Hom}_{\mathcal{B}}(Y', Y)$ with $\text{Im}\, g \leq I_1$, and any $Y' \in \mathcal{Y}$, then $I_2 \in \mathcal{G}_Y$.

Note that T1) can be replaced with an axiom which states that $\mathcal{G}_Y$ is a filter on the lattice $\mathcal{L}_{\mathcal{B}}(H_Y)$.

We know that the map

$$\mathcal{T} \mapsto \mathcal{G}(\mathcal{T}) = \{I \in \mathcal{L}_{\mathcal{B}}(H_Y) \mid Y \in \mathcal{Y},\ H_Y/I \in \mathcal{T}\}$$

establishes a bijection between the hereditary torsion classes and the Gabriel topologies of $\mathcal{B}$, with the inverse

$$\mathcal{G} \mapsto \mathcal{T}(\mathcal{G}) = \{B \in \mathcal{B} \mid \text{Ker}\, g \in \mathcal{G} \text{ for all } g \in \text{Hom}_{\mathcal{B}}(H_Y, B) \text{ and all } Y \in \mathcal{Y}\}.$$

Obviously, this Gabriel topology on $\mathcal{Y}$ offers a better determination of the torsion theory $(\mathcal{T}_{\mathcal{B}}, \mathcal{F}_{\mathcal{B}})$ than the corresponding filter on the lattice $\mathcal{L}_{\mathcal{B}}(S)$ given in 1.8. The following Theorem and Corollary are similar in the spirit, not also in details, to [10, Theorem 4.7, Corollary 4.9 and Theorem 4.10].

2.3. Theorem. *Let $\mathcal{Y}$ be a small preadditive category, $\mathcal{B} = \text{Mod-}\mathcal{Y}$, and $\mathbf{f} : \mathcal{Y} \to \mathcal{A}$ be a functor, where $\mathcal{A}$ is an arbitrary Grothendieck category. Let $(\mathbf{R}, \mathbf{L})$ be adjoint pair induced by $\mathbf{f}$, and $S = \bigoplus_{Y\in\mathcal{Y}} Y$, $U = \mathbf{L}(S)$. Keep the notations of Section 1. If $\mathbf{f}(Y)$ is a projective object of $\mathcal{A}$ for all $Y \in \mathcal{Y}$, and the adjoint pair $(\mathbf{R}, \mathbf{L})$ is right pointed, then $\mathbf{af}(\mathcal{Y}) = \{\mathbf{af}(Y) \mid Y \in \mathcal{Y}\}$ is a set of projective generators for $\mathcal{C}$, and the functor $\mathbf{R}$ restricts to the following equivalences of categories.*

a) $\mathcal{C} \longrightarrow \mathcal{D}$ *with the inverse* $\mathcal{D} \xrightarrow{\mathbf{aL}} \mathcal{C}$;

b) $\text{Pres}[U] \longrightarrow \mathcal{D}$ *with the inverse* $\mathcal{D} \xrightarrow{\mathbf{L}} \text{Pres}[U]$;

c) $\text{GF}[U] \longrightarrow \mathcal{D}$ *with the inverse* $\mathcal{D} \longrightarrow \text{GF}[U]$, $B \mapsto \mathbf{L}(B)/\mathbf{t}_{\mathcal{A}}(\mathbf{L}(B))$.

Moreover, the Gabriel topology on $\mathcal{Y}$ associated with the torsion theory $(\mathcal{T}_{\mathcal{B}}, \mathcal{F}_{\mathcal{B}})$ is given by $\mathcal{G}_Y = \{I \in \mathcal{L}_{\mathcal{B}}(Y) \mid \mathbf{f}(Y) = I\mathbf{f}(Y)\}$ for all $Y \in \mathcal{Y}$, where by $I\mathbf{f}(Y)$ we have denoted the image of the induced morphism $\mathbf{L}(I) \to \mathbf{L}(Y) = \mathbf{f}(Y)$.

Proof. We have seen in paragraph 2.1 that U is a CQF-3 object, and $\mathbf{aL}$ is exact. Observe that $\mathcal{T}_{\mathcal{A}} = \{T \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(\mathbf{f}(Y), T) = 0 \text{ for all } Y \in \mathcal{Y}\}$. We know by [9, Proposition 1.2] that $\mathbf{a}(U)$ is an generator for $\mathcal{C}$. Since $\mathbf{a}$ commutes with direct sums, $\mathbf{a}(U) = \bigoplus_{Y\in\mathcal{Y}} \mathbf{af}(Y)$, hence $\{\mathbf{af}(Y) \mid Y \in \mathcal{Y}\}$ is a set of generators for $\mathcal{C}$. Then, by the generalized Popescu-Gabriel theorem [10, Theorem 4.1], the functor

$$\mathcal{C} \to \text{Mod-}\mathbf{af}(\mathcal{Y}),\ C \mapsto \text{Hom}_{\mathcal{C}}(\mathbf{af}(-), C)$$

is fully-faithful. In order to show $\mathbf{af}(Y)$ is projective in $\mathcal{C}$ for all $Y \in \mathcal{Y}$ observe that a short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$ is determined by an exact sequence

$$0 \to C' \to C \to C'' \to T \to 0$$

in $\mathcal{A}$, with $C', C, C'' \in \mathcal{C}$ and $T \in \mathcal{T}_{\mathcal{A}}$. If we apply the exact functor $\mathrm{Hom}_{\mathcal{A}}(\mathbf{f}(Y), -)$ $(Y \in \mathcal{Y})$, we use the first observation of this proof and the adjunction between $\mathbf{a}$ and $\mathbf{i}$, then the conclusion follows.

For an $Y \in \mathcal{Y}$ and an $A \in \mathcal{A}$, the unit of the adjunction $\nu_A : A \to \mathbf{ia}(A)$ has torsion kernel and cokernel, hence it induces an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathbf{f}(Y), A) \cong \mathrm{Hom}_{\mathcal{A}}(\mathbf{f}(Y), \mathbf{ia}(A)) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{af}(Y), \mathbf{a}(A)),$$

checking that the categories $\mathbf{af}(\mathcal{Y})$ and $\mathbf{f}(\mathcal{Y})$ are equivalent. Since the categories $\mathbf{f}(\mathcal{Y})$ and $\mathcal{Y}$ are equivalent too, so are Mod-$\mathbf{af}(\mathcal{Y})$ and Mod-$\mathcal{Y}$. Then we can easily observe, that the above fully-faithful functor is naturally equivalent to $\mathbf{Ri}$. Therefore to prove a) and b), we apply Proposition 1.6 and Theorem 1.18. To prove c) works the same argument of the proof of Corollary 1.19.

Finally, as in Lemma 1.10, $\mathcal{G}_Y = \{I \in \mathcal{L}_{\mathcal{B}}(Y) \mid \mathbf{f}(Y)/I\mathbf{f}(Y) \in \mathcal{T}_{\mathcal{A}}\}$, for all $Y \in \mathcal{Y}$. Moreover, since $\mathbf{f}(Y)/I\mathbf{f}(Y)$ is isomorphic to $\mathbf{L}(Y/I)$, it belongs to $\mathcal{T}_{\mathcal{A}}$ if and only if it is equal to 0, or equivalently, $\mathbf{f}(Y) = I\mathbf{f}(Y)$. □

2.4. Corollary. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two small preadditive categories, let $\mathbf{f} : \mathcal{Y} \to \mathcal{X}$ be a functor, and $\mathcal{A} =$ Mod-$\mathcal{X}$, $\mathcal{B} =$ Mod-$\mathcal{Y}$. Let $\mathbf{L} : \mathcal{B} \to \mathcal{A}$ be the unique colimits preserving functor which extends $\mathbf{f}$. If $\mathbf{f}$ is fully-faithful, $S = \bigoplus_{Y \in \mathcal{Y}} Y$, and $U = \mathbf{L}(S)$, then the categories $\mathcal{C}$,* Pres$[U]$, GF$[U]$ *and $\mathcal{B}$ are equivalent, where the meaning of symbols $\mathcal{C}$,* Pres$[U]$ *and* GF$[U]$ *is the same as in Section 1.*

Proof. Clearly, $\mathbf{L}$ has a right adjoint, namely

$$\mathbf{R} : \mathcal{A} \to \mathcal{B},\ \mathbf{R}(A) = \mathrm{Hom}_{\mathcal{A}}(H_{\mathbf{f}(-)}, A),$$

where $H_X = \mathrm{Hom}_{\mathcal{X}}(-, X)$ for all $X \in \mathcal{X}$. Note that the Yoneda isomorphism implies $\mathbf{R}(A) \cong A\mathbf{f}$, and $\mathbf{R}$ is a colimits preserving functor, because $H_{\mathbf{f}(Y)}$ is finitely generated projective in $\mathcal{A}$. If $\mathbf{f}$ is fully-faithful, or equivalently, the adjoint pair $(\mathbf{R}, \mathbf{L})$ is right pointed, we deduce, by Theorem 2.3, that the categories $\mathcal{C}$, Pres$[U]$, GF$[U]$ and $\mathbf{R}(\mathcal{A}) = \mathcal{D}$ are equivalent. Moreover, let $B \in \mathcal{B}$ be an object in $\mathcal{B}$ such that $B \in \mathcal{T}_{\mathcal{B}} = \mathrm{Ker}\,\mathbf{L}$. Then there is a presentation of B

$$\bigoplus_{Y'} H_{Y'} \to \bigoplus_{Y} H_Y \to B \to 0,$$

where Y' and Y are chosen among elements, possibly repeated, of $\mathcal{Y}$. Applying the colimits preserving functor $\mathbf{RL}$ to this sequence, and having in mind that $\mathbf{RL}(H_Y) \cong H_Y$ for all $Y \in \mathcal{Y}$ by Yoneda isomorphism, it follows $B \cong \mathbf{RL}(B) \in \mathcal{D}$, hence $\mathcal{D} = \mathcal{B}$. □

2.5. Generalized module category. Fix an arbitrary commutative ring k. By a *k-category* we understand an additive category, whose Hom-sets are equipped with a k-module structure, such that the composition of morphisms is linear in both variables. For example, the concepts of $\mathbb{Z}$-category and preadditive category coincide. Let $\mathcal{X}$ be a skeletally small k-category, and denote by $\mathcal{A} = \text{Mod-}\mathcal{X}$. Clearly, $\mathcal{A}$ is a k-category too. Moreover, $\mathcal{A}$ is a Grothendieck category, with $\mathcal{X}$ as set of finitely generated generators.

Note that $\mathcal{A}$ is equivalent to the category of additive contravariant functors $\mathcal{X}^{\text{op}} \to \text{Mod-}k$. Regarding the small preadditive categories as rings with several objects, the situation is analogous to the case of an ordinary k-algebra R, namely, every R-module becomes automatically a k-module.

In what remains of this paragraph, we consider a particular case, namely, when $\mathcal{X}$ is the (skeletally small preadditive) category of finitely presented R-modules mod-R, where R is a finite dimensional algebra over a field k. Note that $(\text{mod-}R)^{\text{op}} \simeq \text{mod-}R^{\text{op}}$, that is the opposite category of mod-R is equivalent to the category of finitely presented left R-modules, via the usual k-duality functor $\mathbf{D} : \text{mod-}R \to \text{mod-}R^{\text{op}}$, $\mathbf{D}(M) = \text{Hom}_k(M, k)$. We put $\mathcal{A} = \text{Mod-}\mathcal{X}$, that is, $\mathcal{A}$ is the *category of R-generalized modules*. Let M be a finitely presented R-module, and denote $E = \text{End}_R(M)$ and $\mathcal{Y} = \{E\}$. Then $\mathcal{B} = \text{Mod-}\mathcal{Y} = \text{Mod-}E$ is actually the category of the right E-modules. The functor $\mathbf{f} : \mathcal{Y} \to \mathcal{X}$, $\mathbf{f}(E) = M$ is fully-faithful, so it induces a right pointed adjoint pair $(\mathbf{R}, \mathbf{L})$, where $\mathbf{L}$ is the unique colimits preserving functor sending E into $H_M = \text{Hom}_R(-, M)$ (or M after identification), and $\mathbf{R}(A) = \text{Hom}_{\mathcal{A}}(H_M, A) \cong A\mathbf{f}$ for all $A \in \mathcal{A}$. Consequently, $\text{Pres}[H_M]$ is equivalent to Mod-E by Corollary 2.4. This generalizes the equivalences given in [18, Lemma 2.2 and Lemma 2.3]. Indeed, since an equivalence preserves the finitely presented objects, mod-E, is carried into $\text{fp}\,\text{Pres}[H_M]$. On the other hand, since colimits in $\text{Pres}[H_M]$ are computed exactly as in $\mathcal{A}$, $A \in \text{fp}\,\text{Pres}[H_M]$ if and only if A belongs to $\text{Pres}[H_M]$ and it is finitely presented as object of $\mathcal{A}$. Hence

$$\begin{aligned}\text{fp}\,\text{Pres}[H_M] = \{A \in \mathcal{A} \mid &\text{there is an exact sequence} \\ &H_M^m \to H_M^n \to A \to 0, \text{ with } m, n \geq 1\}.\end{aligned}$$

Now, as in [3, proof of Proposition 3.1], the functors belonging to $\text{fp}\,\text{Pres}[H_M]$ are images of a suitable morphism in $\mathcal{A}$ of the form $H_M^n \to \mathbf{D}Hom_R(M, -)^m$, where $m, n \geq 1$ are integers. Moreover, for all $X \in \text{mod-}R$ and all $A \in \text{fp}\,\mathcal{A}$, $A(X) \in \text{mod-}k$, so A can be regarded as a functor $(\text{mod-}R)^{\text{op}} \to \text{mod-}k$. Therefore $\text{fp}\,\text{Pres}[H_M]$ are exactly the functors described in [18, Section 2].

2.6. The category of unital modules over a ring with local units. Recall that a ring (without identity) R is said to be *with enough idempotents* if there is a set $\mathcal{X}$ of pairwise orthogonal idempotents of R, such that

$$R = \bigoplus_{e \in \mathcal{X}} eR = \bigoplus_{e \in \mathcal{X}} Re = \bigoplus_{e \in \mathcal{X}} eRe.$$

A module M over such a ring is said to be *unital* if $MR = M$. With standard arguments, it may be verified that Mod-R, the category of unital modules over R is equivalent to the category of modules over the small preadditive category having as objects the set $\mathcal{X}$, and as morphisms $\mathrm{Hom}_{\mathcal{X}}(e', e) \cong \mathrm{Hom}_R(e'R, eR) \cong eRe'$, the composition of morphisms being induced by the multiplication in R.

Note also that, conversely, if it is given a small preadditive category $\mathcal{X}$, then there is a ring with enough idempotents R such that Mod-$\mathcal{X}$ is equivalent to Mod-R. (see [6, Chapitre II, Proposition 2]).

An associative ring R is called *ring with local units* if it has a set $\mathcal{E}$ of idempotents, such that every finite subset of R is contained in a subring of the form eRe, where $e \in \mathcal{E}$. As in the case of rings with enough idempotents, a module M over a ring with local units is called unital if $MR = M$, and Mod-R denotes the category of unital modules. For $e, e' \in \mathcal{E}$, we define the relation $e \leq e'$ if and only if $ee' = e'e = e$, equivalently, $eRe \subseteq e'Re'$. This is an ordering on $\mathcal{E}$, such that $(\mathcal{E}, \leq)$ is directed, and $R = \sum_{e\in\mathcal{E}} eRe$. Moreover, the R-module M is unital if and only if it is of the form $M = \sum_{e\in\mathcal{E}} Me$, where Me is regarded an abelian subgroup of M. Note also that if R is ring with enough idempotents $\mathcal{X}$, then the set $\{e_1 + \ldots + e_n \mid e_1, \ldots, e_n \in \mathcal{X}\}$ acts a a set of local unit for R. On the other hand, according to [2, p. 12, Remark], the category of unital modules over an arbitrary ring with local units is equivalent to the one of unital modules over a ring with enough idempotents, hence it is a Grothendieck category.

Let R be a ring with local units. Following [2], we say that a unital R-module M is *locally projective* if it is the direct limit of a system $\{M_\lambda \mid \lambda \in \Lambda\}$ of finitely generated, projective direct summands of M, such that $\lambda \leq \lambda'$ whenever the canonical projection $M \to M_\lambda$ factors through $M \to M_{\lambda'}$. Alternatively, we may consider the composite morphisms $M \to M_\lambda \to M$, where $M_\lambda \to M$ are the canonical injections. These morphisms are denoted here by $\epsilon_\lambda \in \mathrm{End}_R(M)$, and they form a set of idempotents of $\mathrm{End}_R(M)$. Thus, the ring $\mathrm{End}_R(M_\lambda)$ is a subring of $\mathrm{End}_R(M)$, actually $\mathrm{End}_R(M_\lambda) \cong \epsilon_\lambda \mathrm{End}_R(M)\epsilon_\lambda$. As in [2, Section 2, p. 11], we construct the ring $E = \mathrm{END}_R(M) = \varinjlim \mathrm{End}_R(M_\lambda)$. Clearly, the set $\{\epsilon_\lambda \mid \lambda \in \Lambda\}$ acts as a set of local units on the ring E. Moreover, $E = \mathrm{End}_R(M)E$, and there is a left E-module structure on M, defined by the restriction of the scalars via the inclusion map $E \to \mathrm{End}_R(M)$. Thus, for all $A \in$ Mod-R, the abelian group $\mathrm{Hom}_R(M, A)$ becomes a E-module, and denote by $\mathrm{HOM}_R(M, A) = \mathrm{Hom}_R(M, A)E$, the largest unitary E-submodule of $\mathrm{Hom}_R(M, A)$. Obviously,

$$\mathrm{HOM}_R(M, A) \cong \varinjlim \mathrm{Hom}_R(M, A)\epsilon_\lambda \cong \varinjlim \mathrm{Hom}_R(M_\lambda, A).$$

In fact, $\mathrm{HOM}_R(M, A)$ consists exactly of those morphisms which factor through a submodule M_λ. Observe that we have just defined a functor

$$\mathrm{HOM}_R(M, -) : \text{Mod-}R \to \text{Mod-}E.$$

Since the usual tensor product can be defined without the use the identity of the ring, we obtain another functor $- \otimes_E M : \text{Mod-}E \to \text{Mod-}R$. For all

$A \in$ Mod-R and all $B \in$ Mod-E, we define the maps

$$\rho_A : \mathrm{HOM}_R(M, A) \otimes_E M \to A,\ \rho_A(f \otimes m) = f(m);$$

$$\sigma_B : B \to \mathrm{HOM}_R(M, B \otimes_E M),\ \sigma_B(b) : m \mapsto b \otimes m.$$

It is routine to verify that ρ_A and σ_B are well defined R, respectively E-morphisms, and they are the counit, respectively unit of the adjunction between $\mathbf{R} = \mathrm{HOM}_R(M, -)$ and $\mathbf{L} = - \otimes_E M$.

Since Mod-R and Mod-E are Grothendieck categories, E is a generator for Mod-E, and $M = E \otimes_E M$, the general settings of paragraph 1.1 are fulfilled. Therefore, we may keep the notations used in Section 1, for the categories $\mathcal{A} =$ Mod-R and $\mathcal{B} =$ Mod-E.

2.7. Proposition. *If $M = \varinjlim M_\lambda$ is a locally projective module over a ring with local units R, then the functor $\mathrm{HOM}_R(M, -)$ restricts to the following equivalences of categories*

a) $\mathcal{C} \longrightarrow \mathcal{B}$ *with the inverse* $\mathcal{B} \overset{\mathbf{a}(-\otimes_E M)}{\longrightarrow} \mathcal{C}$*;*

b) $\mathrm{Pres}[M] \longrightarrow \mathcal{B}$ *with the inverse* $\mathcal{B} \overset{-\otimes_E M}{\longrightarrow} \mathcal{C}$*;*

c) $\mathrm{GF}[M] \longrightarrow \mathcal{B}$ *with the inverse* $\mathcal{B} \longrightarrow \mathrm{GF}[M]$*,* $B \mapsto B \otimes_E M / \mathbf{t}_{\mathcal{A}}(B \otimes_E M)$*.*

Proof. According to [2, p. 12, Remark], the category Mod-E is equivalent to Mod-$\mathrm{End}_E(P)$, where $P = \bigoplus_{\lambda \in \Lambda} M_\lambda$, and $\mathrm{End}_E(P)$ is a ring with enough idempotents. Therefore it is also equivalent to the category Mod-$\mathcal{Y}$, where $\mathcal{Y}$ is a preadditive category, with the set of objects Λ and $\mathrm{Hom}_{\mathcal{Y}}(\lambda', \lambda) \cong \epsilon_\lambda E \epsilon_{\lambda'}$. We denote by $\mathbf{T} :$ Mod-$\mathcal{Y} \to$ Mod-E this last equivalence. Then, obviously, $(\mathbf{T}^{-1}\mathbf{R}, \mathbf{LT})$ is an adjoint pair, which is induced by $\mathbf{f} : \mathcal{Y} \to$ Mod-R, $\mathbf{f}(\lambda) = M_\lambda$. By hypothesis $\mathbf{f}(\lambda)$ is projective in Mod-R, for all $\lambda \in \Lambda$. Moreover, we claim that $\mathrm{Hom}_R(M_{\lambda'}, M_\lambda) \cong \epsilon_\lambda E \epsilon_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda$. Indeed, this is obvious if we replace E with $\mathrm{End}_R(M)$. But for any epimorphism f of M, we have $\epsilon_\lambda f \epsilon_{\lambda'} = \epsilon_\lambda^2 f \epsilon_{\lambda'}^2 \in \epsilon_\lambda E \epsilon_{\lambda'}$, hence our claim holds. This, together with the isomorphism $\mathbf{L}(E_\lambda) = E_\lambda \otimes_E M \cong M_\lambda$ shows that the adjoint pair $(\mathbf{T}^{-1}\mathbf{R}, \mathbf{LT})$ is right pointed. Therefore Theorem 2.3 gives the desired equivalences, but with $\mathcal{D}$ instead of $\mathcal{B}$. The fact that $\mathcal{D} = \mathcal{B}$ follows by [4, Corollary 2.5]. □

The referee communicated us the following alternative proof of the fact that $\mathcal{B} = \mathcal{C}$ in Proposition 2.7, connecting it to the theory in [9]: By [15, Proposition 3.2] and [11, Theorem 1.9] it follows that $\mathcal{B}$ is equivalent to the quotient category of Mod-$\mathrm{End}_R(M)$ modulo the Gabriel filter of right ideals containing E. The argument used in [8, Theorem 1.3] shows that E is the smallest right ideal of $\mathrm{End}_R(M)$ satisfying $EM = M$. On the other hand, this is equivalent to $\mathcal{C}$, according to [9, Theorem 1.7]

2.8. Remark. a) Proposition 2.7 generalizes [2, Theorem 2.5], where the module M is supposed, in addition, to be generator of the category Mod-R.

If this is the case, by Proposition 1.20, we have immediately $\mathcal{C} = \mathrm{Pres}[U] = \mathrm{GF}[U] = \text{Mod-}R$.

b) Remark in [2, p. 12], which plays a crucial role in the proof of Proposition 2.7, is a consequence of the result, generalized by us, stated in [2, Theorem 2.5]. In a preliminary form of this paper, the proof of Proposition 2.7 did not make use of this Remark, and was deduced directly from Corollary 1.19. We preferred this proof, because it is shorter, and, more important, it gives an unitary character to the paper.

2.9. Group graded rings. Let G be a group, $R = \bigoplus_{\gamma \in G} R_\gamma$ a G-graded ring and fix a subgroup H of G, and denote by $H\backslash G$ the set of the right cosets of H in G, and by $[H\backslash G]$ a set of representatives for these cosets. A $H\backslash G$-graded R-module is a R-module N which has a decomposition $N = \bigoplus_{\gamma \in [H\backslash G]} N_{H\gamma}$ as abelian group, such that $N_{H\gamma_1} R_{\gamma_2} \subseteq N_{H\gamma_1\gamma_2}$, for all $\gamma_1 \in [H\backslash G]$ and all $\gamma_2 \in G$. We consider the category $\text{Gr-}(H\backslash G, R)$, whose objects are $H\backslash G$-graded R-modules, and morphisms are grade preserving R-morphisms (see [16] for details). It is well known that this category is a Grothendieck category.

Let M be a G-graded (right) R-module, and $N \in \text{Gr-}(H\backslash G, R)$. Then, forgetting the grade, M becomes a $H\backslash G$-graded R-module, putting $M_{H\gamma} = \bigoplus_{\chi \in H} M_{\chi\gamma}$. Denote by

$$\mathrm{HOM}_{H\backslash G,R}(M, N) = \{f \in \mathrm{Hom}_R(M, N) \mid f(M_{H\gamma}) \subseteq N_{H\gamma} \text{ for all } \gamma \in [H\backslash G]\}.$$

Direct verification shows that $\mathrm{END}_R(M, M) = \mathrm{HOM}_{1\backslash G,R}(M, M)$ is a G-graded ring, M is a G-graded left E, right R-bimodule, and the abelian group $\mathrm{HOM}_{H\backslash G,R}(M, N)$ is actually a $H\backslash G$-graded E-module, for all $N \in \text{Gr-}(H\backslash G, R)$, where E denotes the graded ring $\mathrm{END}_R(M)$. We have defined a functor $\mathrm{HOM}_{H\backslash G,R}(M, -) : \text{Gr-}(H\backslash G, R) \to \text{Gr-}(H\backslash G, E)$, which has a right adjoint, namely $-\otimes_E M : \text{Gr-}(H\backslash G, E) \to \text{Gr-}(H\backslash G, R)$ [16, paragraph 1.10].

Denote by $\mathcal{Y}$ the category whose objects are the cosets $H\backslash G$, with morphisms $\mathrm{Hom}_{\mathcal{Y}}(H\gamma_1, H\gamma_2) \cong E_{\gamma_2^{-1}H\gamma_1}$, and the composition of the morphisms is given by the multiplication in the ring E. In a straightforward manner, we may show that the categories $\text{Gr-}(H\backslash G, E)$ and $(\mathcal{Y}^{\mathrm{op}}, \mathcal{A}b)$ are equivalent, to a $H\backslash G$-graded R-module N corresponding the functor $N^* : \mathcal{Y}^{\mathrm{op}} \to \mathcal{A}b$, given by

$$N^*(H\gamma) = N_{H\gamma} \text{ and } N^*(\alpha)(x) = x\alpha, \text{ for all } \alpha \in E_{\gamma_2^{-1}H\gamma_2}, x \in N_{H\gamma_2},$$

and conversely, to such a functor $N^* : \mathcal{Y}^{\mathrm{op}} \to \mathcal{A}b$ corresponding the $H\backslash G$-graded R-module $N = \bigoplus_{\gamma \in [H\backslash G]} N^*(H_\gamma)$. Note that equivalence between the category of G-graded modules and a category of unital modules over a ring with enough idempotents may be also found in [1, Corollary 2.9].

The functor $- \otimes_E M$ is colimits preserving and extends the functor

$$\mathbf{f} : \mathcal{Y} \to \text{Gr-}(H\backslash G, R), \quad \mathbf{f}(H\gamma) = M(\gamma^{-1}),$$

(this being regarded, as $H\backslash G$-graded), where by $M(\gamma)$ we have denoted the γ-th suspension of M [16, paragraph 1.7]. Indeed, to the object $H\gamma_0 \in \mathcal{Y}$ corresponds the functor $\mathrm{Hom}_{\mathcal{Y}}(-, H\gamma_0) : \mathcal{Y}^{\mathrm{op}} \to \mathcal{A}b$, and by the above isomorphism of categories, this is carried into $\bigoplus_{\gamma\in[H\backslash G]} E_{\gamma_0^{-1}H\gamma} \cong E(\gamma_0^{-1})$. Note that $S = \bigoplus_{\gamma\in G} E(g)$ is a generator for the category Gr-$(H\backslash G, R)$, and $U = S \otimes_E M = \bigoplus_{\gamma\in G} M(\gamma)$. Assume that M is Σ-quasiprojective, which is equivalent, by [16, paragraph 2.16], to the projectivity of M in the full subcategory $\boldsymbol{\sigma}[U]$ of Gr-$(H\backslash G, R)$ consisting of all $H\backslash G$-graded R-modules subgenerated by U. Thus Theorem 2.3, gives as a particular case [16, Theorem 3.9].

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References

[1] Albu, T., *Pure ideals, quotient categories and infinite group graded rings*, Comm. Algebra, **15** (1990), 839-862.

[2] Anh, P. M. and Márki, L., *Morita equivalence for rings without identity*, Tsukuba J. Math., **11** (1987), 1-16.

[3] Auslander, M. and Reiten, I., *Stable equivalence of dualizing R-varieties*, Adv. Math, **12** (1974), 306-366.

[4] Castaño Iglesias, F., Gómez Torrecillas, J. and Wisbauer, R., *Adjoint functors and equivalences*, preprint, available at `http://reh.math.uni-duesseldorf.de/~wisbauer/`.

[5] Fuller, K. R., *Density and equivalence*, J. Algebra, **29** (1974), 528-550.

[6] Gabriel, P., *Des catégories abeliennes*, Bull. Soc. Math. France, **90** (1962), 323-448.

[7] García Hernández, J.L. and Gómez Pardo, J.L., *On endomorphism rings of quasiprojective modules*, Math. Z., **196** (1987), 87-108.

[8] García Hernández, J.L. and Gómez Pardo, J.L., *Self-injective and PF-endomorphism rings*, Israel J. Math., **58** (1987), 324-350.

[9] García, J.L. and Saorín, M., *Endomorphism rings and category equivalences*, J. Algebra, **127** (1989), 182-205.

[10] Garkusha, G. A., *Grothendieck categories*, to appear in St. Petersburg Math. J., **13** (2002).

[11] Gentle, R., *T.T.F. Theories in abelian categories*, Comm. Alg., **16** (1988), 877-908.

[12] Gómez Pardo, J.L. and Martínez Hernández, J., *Coherence of endomorphism rings*, Arch. Math., **48** (1987), 40-52.

[13] Harada, M., *Perfect categories IV*, Osaka J. Math., **10** (1973), 585-596.

[14] Krause, H., *The spectrum of a module category*, Habilitationsschrift, Universität Bielefeld, 1998.

[15] Leeney, M. G., *LLI Rings and modules of type LI*, Comm. Alg., **20** (1992), 943-953.

[16] Marcus, A. and Modoi, C., *Graded endomorphism rings and equivalences*, preprint available at `http://math.ubbcluj.ro/~marcus/` .
[17] Mitchell, B., *Theory of Categories*, Academic Press, New-York, 1965.
[18] Okuyama, T. and Uno, K., *On the vertices of modules in the Auslander-Reiten quiver II*, Math. Z., **217** (1994), 121-141.
[19] Popescu, N. and Popescu, L., *Theory of categories*, Editura Academiei, Bucureşti, România and Sijthoff & Noordhoff International Publishers, 1979.
[20] Stenström, B., *Rings of Quotients*, Springer-Verlag, 1975.
[21] Wisbauer, R., *Static modules and equivalences*, Interactions between ring theory and representations of algebras (Murcia), 423–449, Lecture Notes in Pure and Appl. Math., 210, Dekker, New York, 2000.

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